B Computing the Model With Two-Sided Limited Commitment

To compute this model, we have to solve two tasks:

1. Compute the value functions $F(v, z, z_m)$ and $V(z, z_m)$ conditional no a given vector of outside options $V^u(z)$.

2. Find the vector $V^u(z)$ that satisfies Equ. (11).

Part 2. of the problem can be done by a standard quasi-Newton algorithm such as Broyden's method. In the following I describe how to solve Part 1., how to solve for the optimal contract conditional on $V^u(z)$.

B.1 First order conditions of the problem of the principal (the firm)

The Lagrangian to solve this problem is

$$\begin{aligned}\mathcal{L} = Y(z, z_m) - w + \beta\,\mathrm{E}_{z'}(1-\delta)F\left(V\left(z', z_m\right), z', z_m\right) \\ + \lambda\left[-v + u(w) + \beta\,\mathrm{E}_{z'}\left[(1-\delta)V\left(z', z_m\right) + \delta V^u(z')\right]\right] \\ + \sum_i \mu_i\left[V\left(z_i', z_m\right) - V^u(z_i')\right] \\ + \sum_i \nu_i\left[F\left(V\left(z_i', z_m\right), z_i'\right)\right]\end{aligned}$$

The first order conditions are

$$\frac{\partial\mathcal{L}}{\partial w} = -1 + \lambda u'(w) = 0 \tag{31a}$$

$$\frac{\partial\mathcal{L}}{\partial V\left(z_i'\right)}\frac{1}{Pr\left\{z_i'|z\right\}} = \left(\beta(1-\delta) + \nu_i\right)F'\left(V\left(z', z_m\right), z', z_m\right) + \lambda\beta(1-\delta) + \mu_i \tag{31b}$$

From (31a) we get

$$\lambda = \frac{1}{u'(w)} > 0 \tag{32}$$

If none of the inequality constraints on $V\left(z_i'\right)$ is binding, we get from (31b) that

$$F'\left(V\left(z', z_m\right), z', z_m\right) = -\lambda < 0 \tag{33}$$

Combining (32) and (33) we get

$$F'\left(V\left(z', z_m\right), z', z_m\right) = -\frac{1}{u'(w)} \tag{34}$$

We know that $F(V(z', z_m), z', z_m)$ is strictly concave in $V(z', z_m)$, therefore $F''(V(z', z_m), z', z_m)\, 0$. This means that a higher promised value $V(z', z_m)$ implies a lower derivative $F'(V(z', z_m), z', z_m)$. From the rhs of (34) and the concavity of u, we then get that a higher promised value is related to a higher current wage. To give a higher value to a worker, the firm increases both the current wage and the values promised for the next period.

B.2 Recursive computation of the optimal contract

We solve the principal's problem by dynamic programming, and use the first order conditions from the last subsection to design a fast algorithm. Part of the problem is to compute the feasible set of promised values for next period. This feasible set is determined by the constraints from limited commitment, Equs. (6d) and (6e).

Solving the model backward in time, we can easily determine the feasible set of values $V(z)$, given any z, in the last period

$$\underline{V}_z^0 = u(z), \qquad \overline{V}_z^0 = V^u(z) \tag{35}$$

Now assume that $\underline{V}_{z_i}^{t+1}$ and $\overline{V}_{z_i}^{t+1}$ are given and $F_{t+1}(V(z), z, z_m)$ is known.

For a given z_i, we trace out $F_t(v, z, z_m)$ for all values of v. To find a point on the graph of this function, choose any possible wage $\tilde{w}$. Then, for all z_i', we get from (34) that

$$\tilde{V}(z_i'; \tilde{w}) = \min\left\{\max\left\{F'^{-1}\left(-\frac{1}{u'(w)}, z_i'\right) \underline{V}_{z_i'}^{t+1}\right\}, \overline{V}_{z_i'}^{t+1}\right\} \tag{36}$$

Then we get the corresponding

$$F_t(z_i, \tilde{w}) = z_i - \tilde{w} + \beta \sum_j \pi_{i,j}(1-\delta) F_{t+1}\left(\tilde{V}(z', z_m; w), z', z_m\right) \tag{37}$$

and

$$V_t(z_i, \tilde{w}) \, = u(\tilde{w}) + \beta \sum_j \pi_{i,j}(1-\delta) \tilde{V}(z', z_m; w) \tag{38}$$

To each chosen $\tilde{w}$, we get a point $V_t(z_i, \tilde{w}), F_t(z_i, \tilde{w})$ on the graph of the value function. We approximate this function then by some kind of spline approximation based on the function values at those points. The fact that the points are irregular in the space of v-values is no problem for spline approximations.

In more detail, we implement the following procedure: find

$$F'_{min} \equiv \min_i F'_{t+1}\left(\overline{V}_{z_i}^{t+1}, z_i\right) \tag{39}$$

$$F'_{max} \equiv \max_i F'_{t+1}\left(\underline{V}_{z_i}^{t+1}, z_i\right) \tag{40}$$

Using (34), compute the corresponding wages:

$$w^{max} = u'^{-1}\left(-1/F'_{min}\right)$$
$$w^{min} = u'^{-1}\left(-1/F'_{max}\right)$$

Now choose a grid of points w_j, $j = 1, \ldots, n_w$, with $w_1 = w^{min}$ and $w_{n_w} = w^{min}$.

Notice that for any $v > F_t(z_i, w^{max})$ we get

$$F(v, z_i) = F_t(z_i, w^{max}) + w^*(v) - w^{max} \tag{41a}$$

where $w^*(v)$ is defined such that

$$u(w^*(v)) - u(w^{max}) = v - F_t(z_i, w^{max}) \tag{41b}$$

and analogously for $v < F_t(z_i, w^{min})$. This means, if we have traced out the V and F for the grid of wages defined above, and approximate $V(F)$ within this grid by a spline, we can compute the function outside the grid using (41).