Notes on the Mortensen/Pissarides Model

Michael Reiter, IHS, Vienna

4 MP Model in discrete time

Basic source: Pissarides (1990, Chapter1)

Notation	Meaning
u	unemployment rate
v	number of vacancies
f	job finding rate
q	vacancy filling rate
θ	market tightness
V^W	value employed
V^U	value unemployed
Σ^W	worker surplus
Σ	total surplus
b	unemployment utility (benefit)
m	total number of new matches
σ	separaton rate
λ	exponent of u in matching function
α	worker bargaining power

4.1 The model

Mechanics of unemployment dynamics

:

- Dynamic equation for unemployment"

$$u_t = u_{t-1} + (1-\sigma)(1-u_{t-1} - m_{t-1} \tag{1}$$

- Matching function:

$$m_t = \mathcal{M}(u_t, v_t) \tag{2}$$

Equal chance of finding a job:

$$f_t = \frac{m_t}{u_t} \tag{3}$$

Equal chance of filling a vacancy:

$$q_t = \frac{m_t}{vt} \tag{4}$$

Properties of the matching function:

- We assume constant returns to scale
- This implies

$$f_t = f(\theta_t), \qquad q_t = q(\theta_t) \tag{5}$$

  where market tightness θ is defined as

$$\theta_t = \frac{v_t}{ut} \tag{6}$$

- f is increasing in θ, q is decreasing in θ, and

$$f_t = \theta_t q_t \tag{7}$$

Time line:

- Beginning of period t: aggregate shock to labor productivity p_t realizes
- the employed workers $(1-u_t)$produce
- firms post vacancies v_t
- matching and separation (simultaneously):
  * employed workers separate with probability σ
  * unemployed workers and vacancies generate m_t matches
- Important example: Cobb-Douglas matching function

$$\mathcal{M}(u,v) = \phi u^\lambda v^{1-\lambda} \tag{8}$$

  such that, for example, $f = \phi\lambda\theta^{1-\lambda}$. Notice that this formulation does not guarantee $f \leq 1$. In setting up the model, we have to choose the time period small enough so that $f \leq 1$ is satisfied with probability 1.

- Stationary state u^*: from $u^* = u^* + (1-\sigma)(1-u^* - m^*$ we get

$$u^* = \frac{\sigma}{\sigma + f^*} \tag{9}$$

Job creation

:

- Posting a vacancy costs κ (or $\kappa \cdot p_t$ as in Pissarides, Chapter 1).
- If a vacancy is not filled in this period, it disappears, nothing valuable lef
- free entry (unlimited supply of potential vacancies); equilibrium condition therefore

$$\kappa = q_t \beta \operatorname{E}_t J_{t+1} \tag{10}$$

  where J_t is the value of a filled job, satisfying the recursive equation

$$J_t = p_t - w_t + (1-\sigma)\beta \operatorname{E}_t J_{t+1} \tag{11}$$

Discrete versus continuous time

Without aggregate shocks, formulas look somewhat simpler in discrete time. To go from discrete time, we denote by Δ the length of the time period, and then let $\Delta \to 0$. All flows are transition rates are now understood as rates per time unit. The time discount factor is written as $\beta = (1 - r\Delta)$.

Then the value of a filled job satisfies

$$J_t = (p_t - w_t)\Delta + (1 - \sigma\Delta)(1 - r\Delta)J_{t+\Delta} \tag{12}$$

For very small Δ, we can approximate $J_{t+\Delta} = J_t + \dot{J}_t\Delta$ where $\dot{J}_t\Delta$ denotes the time derivative. Neglecting terms in Δ^2,

Equ.(14) then simplifies to

$$rJ_t\Delta = (p_t - w_t)\Delta - \sigma\Delta J_t + \dot{J}_t\Delta) \tag{13}$$

Dividing by Δ gives

$$rJ_t = (p_t - w_t) - \sigma J_t + \dot{J}_t) \tag{14}$$

In a stationary state, $\dot{J}_t = 0$ and we get

$$J^* = \frac{p^* - w^*}{r + \sigma} \tag{15}$$

Worker values

Key assumptions:

- Linear utility of workers
- No saving
- Same discount rate as firms, β
- Firm profits are not accounted for (assume that firm owners are different people, not modeled here).

$$V_t^W = w_t + \beta\,\mathrm{E}_t\left[(1-\sigma)V_{t+1}^W + \sigma V_{t+1}^U\right] \tag{16}$$
$$V_t^U = b + \beta\,\mathrm{E}_t\left[f_t V_{t+1}^W + (1-f_t)V_{t+1}^U\right] \tag{17}$$

Worker surplus, defined as $\Sigma^W \equiv V_t^W - V_t^U$, then follows

$$\Sigma_t^W = w_t - b + \beta\,\mathrm{E}_t\left[(1-\sigma-f_t)\Sigma_{t+1}^W\right] \tag{18}$$

Stationary state:

$$\Sigma^{W*} = \frac{w^* - b}{1 - \beta(1 - \sigma - f^*)} \tag{19}$$

In continuous time steady state:

$$\Sigma^{W*} = \frac{w^* - b}{r + \sigma + f^*} \tag{20}$$

Wage formation

Both sides earn an economic rent from the match:

- Workers do not have to look for a job
- Firms earn expected profits; they have nothing left if the job is separated

It is theoretically unclear how these rents are split between the two sides. We assume "generalized Nash bargaining": the total surplus $\Sigma_t \equiv \Sigma^W - t + J_t$ is split according to fixed shares:

$$\frac{\Sigma_t^W}{J_t} = \frac{\alpha}{1-\alpha} \tag{21}$$

This implies (add (14) and (17) and subtract (17))

$$\Sigma_t = p_t - b + \beta \operatorname{E}_t \left[(1-\sigma-f_t)\alpha\Sigma_{t+1} + (1-\sigma)(1-\alpha)\Sigma_{t+1}\right] = p_t - b + \beta \operatorname{E}_t \left[(1-\sigma-\alpha f_t)\alpha\Sigma_{t+1}\right] \tag{22}$$

In continuous time steady state:

$$\Sigma^* = \frac{p^* - b}{r + \sigma + \alpha f^*} \tag{23}$$

Wage in steady state is follows from

$$(1-\alpha)\frac{w^* - b}{r + \sigma + f^*} = \alpha \frac{p^* - w^*}{r + \sigma} \tag{24}$$

This gives

$$\frac{w^* - b}{p^* - w^*} = \frac{\alpha}{1-\alpha} \frac{r + \sigma + f^*}{r + \sigma} \tag{25}$$

The first fraction on the rhs of (25) reflects the bargaining power, the second fraction reflects the outside options (you can call it "bargaining position") of worker and firm.

Conclusions:

- For given f, $w \to p$ when $\alpha \to 1$ and $w \to b$ when $\alpha \to 0$.
- For $\alpha = 1/2$, wage is much closer to p than to b, because f is much higher then $\sigma + r$.

Using

$$f^* = \theta^* \frac{\kappa}{J^*} = \theta^* \kappa \frac{r + \sigma}{p^* - w^*} \tag{26}$$

we get

$$w^* - b = \frac{\alpha}{1-\alpha} \left[p^* - w^* + \kappa\theta^*\right] \tag{27}$$

This equation defines the wage as a linear and positive function of tightness. This is called the **wage curve**.

The **Job creation curve** is obtained by solving the free entry condition in steady state $\kappa = q(\theta^*)\frac{p^*-w^*+}{r+\sigma}$ for the wage as a function θ. This function is negatively sloped. The intersection of the wage curve and the job creation curve determines tightness and the wage.

The **Beveridge curve** is a steady state relationship between the unemployment rate and the number of vacancies. It follows from $u^* = \frac{\sigma}{\sigma+f^*}$ and is negatively sloped. With Cobb-Douglas matching function this becomes

$$u = \frac{\sigma}{\sigma + \lambda\phi\theta^{1-\lambda}} \tag{28}$$

Empirically, the negative relationship holds very well over the business cycle as well (at least in the US), because the unemployment rate adjusts very quickly to the level given by (9).

Does the Beveridge curve hold in the model (not just in steady state, but over the business cycle)?

- If the fluctuations are driven by shocks to labor productivity, a positive shock increases vacancy formation, and unemployment then goes down as a consequence. This generates a negative correlation between u_t and v_t, as seen in the data.
- If the fluctuations are generated by shocks to the separation rate σ, a positive shock increases the unemployment rate, this increases the vacancy filling probability for firms, and because of the free entry condition, firms create more vacancies. This generates a positive correlation between u_t and v_t, contrary to the data, and is a main reason why most models treat the separation rate as constant.

4.2 Efficiency of the decentralized solution

Write the planner problem as

$$\max_{v_t,u_t} \mathrm{E} \sum_{t=0}^{\infty} \beta^t \left[bu_t + z_t(1-u_t) - \kappa v_t\right] \tag{29}$$

subject to

$$u_t = u_{t-1} + \sigma(1-u_{t-1}) - \mathcal{M}(u_{t-1}, v_{t-1}) \tag{30}$$

This can be solved using the Lagrangian

$$\mathcal{L} = \mathrm{E} \sum_{t=0}^{\infty} \beta^t \left[bu_t + z_t(1-u_t) - \kappa v_t - \mu_t \left(-u_t + u_{t-1} + \sigma(1-u_{t-1}) - \mathcal{M}(u_{t-1}, v_{t-1})\right)\right] \tag{31}$$

The first order conditions are

$$\beta^{-\tau}\frac{\partial \mathcal{L}}{\partial v_\tau} = -\kappa + \beta\,\mathrm{E}_\tau\,\mu_{\tau+1}\frac{\partial \mathcal{M}(u_\tau, v_\tau)}{\partial v_\tau} = 0 \tag{32}$$

$$\beta^{-\tau}\frac{\partial \mathcal{L}}{\partial u_\tau} = b - p_\tau + \mu_\tau - \beta\,\mathrm{E}_\tau\,\mu_{t+1}\left[\left(+1 - \sigma(1-u_\tau) - \frac{\partial \mathcal{M}(u_\tau, v_\tau)}{\partial u_\tau}\right)\right] = 0 \tag{33}$$

Using the matching function (8), this becomes

$$\kappa = \beta \, \mathrm{E}_\tau \, \mu_{\tau+1}(1-\lambda)\phi\theta_\tau^{-\lambda} \tag{34a}$$

$$\mu_\tau = b - p_\tau + \beta \, \mathrm{E}_\tau \, \mu_{t+1} \left[\left(1 - \sigma - \lambda\phi\theta^{1-\lambda} \right) \right] = 0 \tag{34b}$$

Notice that the job creation condition and the surplus condition can be written as

$$\kappa = \beta q_\tau \, \mathrm{E}_\tau (1-\alpha)\Sigma_{\tau+1} = \beta \, \mathrm{E}_\tau \, \Sigma_{\tau+1}(1-\alpha)\phi\theta_\tau^{-\lambda} \tag{35a}$$

$$\Sigma_\tau = p_\tau - b + \beta \, \mathrm{E}_\tau \, \Sigma_{t+1} \left[\left(1 - \sigma - \alpha\phi\theta^{1-\lambda} \right) \right] = p_\tau - b + \beta \, \mathrm{E}_\tau \, \Sigma_{t+1} \left[\left(1 - \sigma - \alpha\phi\theta^{1-\lambda} \right) \right] \tag{35b}$$

We see that the equation systems (34) and (35) are equivalent if we identify the Lagrange multipler μ with the total surplus of the dencetralized economy, and under the condition

$$\alpha = \lambda \tag{36}$$

(36) is called "Hosios condition" after Hosios (1990).

Interpretation: vacancy positing involves two externalities:

1. it increases the job finding probability of unemployed workers; this is a positive externality, since the firm only cares about its own surplus, which equals $(1-\alpha)\Sigma$, rather than the total surplus Σ.
2. it decreases the vacancy filling probability of other firms this is a negative externality, because firms consider their own filling probability $\phi\theta^{-\lambda}$, not taking into account that the effect of vacancies on employment is gien by $(1-\lambda)\phi\theta^{-\lambda}$.

If the the matching function is Cobb-Douglas, and the Hosios condition (36) is satisfied, these two externalities exactly cancel. Then the decentralized equilibrium is Pareto efficient (considering that the planner is subject to the same mathing friction as private agents.

It is not clear why $\alpha = \lambda$ should hold in reality. These are two completely different parameters. If $\alpha > \lambda$, workers ask for a too high wage when they are matched. Firms anticipate this, and post few vacancies, so that unemployment is too high. There is no competitive mechanism here that would drive wages down. To guarantee efficiency, one has to modify the model so that firms post a wage and are committed to stick to it (Moen 1997). This model then has the same dynamics as the MP model under the Hosios condition.

4.3 Unemployment fluctuations and the role of the surplus

With standard parameters (such as $b = 0.4p$), and business cycles driven by productivity shocks, it turns out that the MP model generates very low unemployment fluctuations (Shimer 2005; Costain and Reiter 2008). This is not surprising: we have seen that unemployment fluctuations are efficient in the MP model, and if it is mucn better having a job than not having a job, a slight decrease in productivity does not mean that many more people should stay unemployed. The efficient response is to lower the wage in about the same proportion as productivity, and decrease vacancies (and therefore increase unemployment) by a small amount only.

Unemployment fluctuations in the data are much higher. There are two main ways of modifying the model so as to generatre larger fluctuations:

- If wages are "sticky": if wages cannot move much, for whatever reason, after a change in productivity, then the incentives to create vaacancies change a lot, generating large fluctuations in uemploment. These fluctuations are inefficient (as they most likely are in the real world).
- If the surplus from having a job is small (b not much smaller than p), then a small change in productivity Example: if $b = 0.95p$, a 6 percent reduction in productivity would already mean that nobody should work. Such a calibration is proposed in Hagedorn and Manovskii (2008). It is not obviously wrong, because b includes both unemployment benefits and the value of leisure and home production. (A counter-argument is given in Costain and Reiter (2008), cf. below).

To see the effect of the surplus more formally, use the Job creation condition to get

$$\kappa = \beta\theta^{-\alpha}\frac{p-b}{r+\sigma+\alpha\theta^{1-\alpha}} \approx \beta\theta^{-\alpha}\frac{p-b}{\alpha\theta^{1-\alpha}} = \beta\frac{p-b}{\alpha\theta} \tag{37}$$

The $\approx$ follows from the fact that, in a reasonably calibrated model $\theta^{1-\alpha} = f >> r+\sigma$. This implies that

$$d\log\theta \approx d\log(p-b) \tag{38}$$

The smaller is $p-b$, the larger is a change in p or in b on the log (or percentage) change of $p-b$, and the bigger is the effect on market tightness and unemployment.

Notice that the effects of p and b are symmetric. Costain and Reiter (2008) argue that a value of b that is so large that unemployment fluctuations are high, implies an unrealistically strong long-run effect of changes in b on the unemployment rate.

References

Costain, J. and M. Reiter (2008). Business cycles, unemployment insurance, and the calibration of matching models. *Journal of Economic Dynamics and Control 32*(4), 1120–55.

Hagedorn, M. and I. Manovskii (2008). The cyclical behavior of equilibrium unemployment and vacancies revisited. *American Economic Review 98*(4), 1692–1706.

Hosios, A. J. (1990). On the efficiency of matching and related models of search and unemployment. *Review of Economic Studies 57*(2), 279–98.

Moen, E. R. (1997). Competitive search equilibrium. *Journal of Political Economy 105*, 385–411.

Pissarides, C. A. (1990). *Equilibrium Unemployment Theory.* Oxford: Basil Blackwell.

Shimer, R. (2005). The cyclical behavior of equilibrium unemployment and vacancies. *American Economic Review 95*, 25–49.