Online-Appendix

"State Reduction and Second-order Perturbations of Heterogeneous Agent Models"

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B The distribution of the i.i.d. Shocks

B.1 Motivation

Part of the methodology outlined in Section 3 is to assume that all agents receive a shock in each period that is i.i.d. across time and agents. The main methodological purpose of the i.i.d. shocks is to smooth the value function as well as the transition function of the cross-sectional distribution. The shock should have a sufficiently smooth density function, denoted by ϕ, but should also have bounded support $[\underline{\xi}, \bar{\xi}]$, in order to preserve the sparsity of the state transition equations.

To motivate our construction of the shock in Section B.2, consider that the value function is interpolated by a cubic spline in the continuous state x, which is twice differentiable everywhere, it is desirable that the theoretical value function is twice differentiable in x. Now consider any function $g(x, \xi, y)$, where x can be a continuous idiosyncratic variable or an aggregate variable, and y is a function of (x, ξ). We assume that g is smooth in its 3 arguments, but y may be non-differentiable, or even discontinuous, for a finite number of ξ-values. Define $\mathcal{I}(x) \equiv \int_{\underline{\xi}}^{\bar{\xi}} g(x, \xi, y(x, \xi))\phi(\xi)\, d\xi$. Assume that, for a given x, $y(x, \xi)$ is given in the open interval $(\underline{\xi}, \xi^*(x))$ by an three times differentiable function y^l and in the open interval $(\xi^*(x), \bar{\xi})$ by the three times differentiable function y^r.

$$\mathcal{I}(x) = \int_{\underline{\xi}}^{\xi^*(x)} g(x, \xi, y^l(x, \xi))\phi(\xi)\, d\xi + \int_{\xi^*(x)}^{\bar{\xi}} g(x, \xi, y^r(x, \xi))\phi(\xi)\, d\xi \tag{48}$$

Formally differentiating with respect to x gives

$$\begin{aligned}\mathcal{I}'(x) = \int_{\underline{\xi}}^{\xi^*(x)} & g_x(x,\xi,y^l(x,\xi)) + g_y(x,\xi,y^l(x,\xi))y^l_x(x,\xi)\phi(\xi)\,d\xi \\ & + \int_{\xi^*(x)}^{\bar{\xi}} g_x(x,\xi,y^r(x,\xi)) + g_y(x,\xi,y^r(x,\xi))y^r_x(x,\xi)\phi(\xi)\,d\xi \\ & \qquad + \frac{d\,\xi^*(x)}{d\,x}\left[g(x,\xi^*,y^l(x,\xi^*)) - g(x,\xi^*,y^r(x,\xi^*))\right]\phi(\xi^*) \end{aligned} \tag{49}$$

Differentiating again with respect to x gives

$$\mathcal{I}''(x) = \ldots + \left(\frac{d\,\xi^*(x)}{d\,x}\right)^2 \phi'(\xi^*)\left[g^l(x,\xi^*,y(x,\xi^*)) - g^r(x,\xi^*,y(x,\xi^*))\right] \tag{50}$$

If $g(x,\xi^*,y^l(x,\xi^*)) \neq g(x,\xi^*,y^r(x,\xi^*))$, (50) shows that the second derivative of $\mathcal{I}(x)$ rests on the differentiability of ϕ. Similarly, one can show that the third derivative of $\mathcal{I}(x)$ rests on the second derivative of ϕ. The next section will construct a specific twice differentiable density ϕ.

B.2 The density function

The following constructs a shock distribution with the following properties:

1. symmetric around zero
2. finite support $(-\bar{\xi},\bar{\xi})$
3. piece-wise polynomial density function that is twice differentiable everywhere
4. normalized to unit variance.

The density function ϕ is constructed by two forth-degree polynomials, sliced together at $\xi = 0$. There are 11 free parameters, namely two times 5 parameters of the polynomial, and $\bar{\xi}$. They are determined by 11 conditions:

- the value, first and second derivative are all zero at $\xi = -\bar{\xi}$ and at $\xi = \bar{\xi}$
- the value, first and second derivative are equal fore the two polynomials at $\xi = 0$
- the density integrates to one: $\int_{-\bar{\xi}}^{\bar{\xi}} \phi(\xi)\,d\xi = 1$

- the shock has unit variance $\int_{-\bar{\xi}}^{\bar{\xi}} \xi^2 \phi(\xi)\, d\xi = 1$

The expected value of zero arises from the symmetry of the construction and need not be imposed. By symmetry, the distribution has zero skewness. It has a kurtosis of 2.625, lower than the skewness of 3 of the normal distribution. This is a consequence of the bounded support.

B.3 Approximating expected values

To compute the expected value $\int g(\xi)\phi(\xi)\, d\xi$ of any function $g(\xi)$, I interpolate $g(\xi)$ piecewise linearly on a grid of knot points $-\bar{\xi} = \bar{\xi}_1 < \bar{\xi}_2, \ldots, \bar{\xi}_{n-1} < \bar{\xi}_{n_\xi} = \bar{\xi}$. Piecewise linear interpolation, unlike higher-order interpolation, guarantees that the expected value is monotonically increasing in the function values $f(\bar{\xi}_i)$. The expected value of the interpolated function can be written as $\sum_{i=1}^{n_\xi} \omega_i f(\bar{\xi}_i)$, where the weights ω_i are positive and independent of f.

For a second-order perturbation, second derivatives of all quantities in the model are needed. Let us now assume that the function f depends on parameters α and β, which might be aggregate variables in the model. For notational brevity, we suppress α and β as arguments of $g(\xi)$. It is natural to approximate

$$\frac{\partial \int g(\xi)\phi(\xi)\, d\xi}{\partial \alpha} \approx \sum_{i=1}^{n_\xi} \omega_i \frac{\partial g(\bar{\xi}_i)}{\partial \alpha}, \qquad \frac{\partial^2 \int g(\xi)\phi(\xi)\, d\xi}{\partial \alpha \partial \beta} \approx \sum_{i=1}^{n_\xi} \omega_i \frac{\partial^2 g(\bar{\xi}_i)}{\partial \alpha \partial \beta} \tag{51}$$

This gives a valid approximation if $g(\xi)$ is such that $\frac{\partial \int g(\xi)\phi(\xi)\, d\xi}{\partial \alpha} = \int \frac{\partial g(\xi)}{\partial \alpha} \phi(\xi)\, d\xi$ and $\frac{\partial^2 \int g(\xi)\phi(\xi)\, d\xi}{\partial \alpha \partial \beta} = \int \frac{\partial^2 g(\xi)}{\partial \alpha \partial \beta} \phi(\xi)\, d\xi$ which requires that $\frac{\partial g(\xi)}{\partial \alpha}$ and $\frac{\partial^2 g(\xi)}{\partial \alpha \partial \beta}$ are continuous in ξ.

If the value, first derivative or second derivative of f is discontinuous at a threshold point $\xi = \hat{\xi}$, we can write the integral as

$$\int g(\xi)\phi(\xi)\, d\xi = \int_{-\bar{\xi}}^{\hat{\xi}} g_1(\xi)\, \phi(\xi)\, d\xi + \int_{\hat{\xi}}^{\bar{\xi}} g_2(\xi)\, \phi(\xi)\, d\xi \tag{52}$$

where the second derivatives of both $g_1(\xi)$ and $g_2(\xi)$ are continuous. Taking into account that the threshold point $\hat{\xi}$ can itself depend on parameters, the derivatives are given by

$$\frac{\partial \int g(\xi)\phi(\xi)\, d\xi}{\partial \alpha} = \int_{-\bar{\xi}}^{\hat{\xi}} \frac{\partial g_1(\xi)}{\partial \alpha}\, \phi(\xi)\, d\xi + \int_{\hat{\xi}}^{\bar{\xi}} \frac{\partial g_2(\xi)}{\partial \alpha}\, \phi(\xi)\, d\xi + \frac{\partial x}{\partial \alpha}(g_1(\hat{\xi}) - g_2(\hat{\xi})) \tag{53}$$

and

$$\frac{\partial^2 \int g(\xi)\phi(\xi)\,d\xi}{\partial\alpha\partial\beta} = \int_{-\bar{\xi}}^{\hat{\xi}} \frac{\partial^2 g_1(\xi)}{\partial\alpha\partial\beta}\,\phi(\xi)\,d\xi + \int_{\hat{\xi}}^{\bar{\xi}} \frac{\partial^2 g_2(\xi)}{\partial\alpha\partial\beta}\,\phi(\xi)\,d\xi + \frac{\partial x}{\partial\alpha}\frac{\partial x}{\partial\beta}\cdot(g_1'(\hat{\xi}) - g_2'(\hat{\xi}))$$

$$+ \frac{\partial x}{\partial\beta}\cdot\left(\frac{\partial g_1(\hat{\xi})}{\partial\alpha} - \frac{\partial g_2(\hat{\xi})}{\partial\alpha}\right) + \frac{\partial x}{\partial\alpha}\cdot\left(\frac{\partial g_1(\hat{\xi})}{\partial\beta} - \frac{\partial g_2(\hat{\xi})}{\partial\beta}\right) + \frac{\partial^2 x}{\partial\alpha\partial\beta}\cdot(g_1(\hat{\xi}) - g_2(\hat{\xi})) \quad (54)$$

A valid perturbation approach therefore requires to compute the threshold point as well as its first and second derivatives. The integrals $\int_{-\bar{\xi}}^{\hat{\xi}} \frac{\partial^2 g_1(\xi)}{\partial\alpha\partial\beta}\,\phi(\xi)\,d\xi$ etc. will again be evaluated by piecewise linear interpolation of $g_1(\xi)$. For this purpose, the threshold point $\hat{\xi}$ is inserted into the grid $\bar{\xi}_1, \ldots, \bar{\xi}_{n_\xi}$.

B.4 Differentiating threshold points

In a perturbation solution, the reaction of individual policies to aggregate shocks comes from two sources. First, the derivatives of the continuous policy function with respect to the aggregate states, obtained either from differentiating the first order conditions (case of interior optimum) or from differentiating the boundary conditions (in case of binding constraints). Second, the derivatives of critical points (called "threshold points" in the following), where behavior changes from one regime to another one. For example, the labor supply in the indivisible-labor model is exclusively driven by the movements of the threshold point from working to non-working.

Individual policy functions depend on the state vector (aggregate and idiosyncratic), computed on a finite grid of states, and on the individual i.i.d. shock ξ, which is treated as a continuous variable. At any point on the grid of individual states, a threshold point is a point $\xi = \hat{\xi}$ where the solution regime changes. A perturbation solution requires to compute the response of switch points to changes in the aggregate variables. I consider four different types of threshold points:

1. Hitting a boundary constraint of the endogenous state (for example a non-negativity constraint on the end-of-period state).
2. Hitting a boundary constraint of another endogenous variable (such as the non-negativity constraint on hours worked).
3. Changing the discrete choice, such as working versus not working in the model of indivisible labor.

4. A discontinuity of the continuous choice variables, arising when the global maximum switches from one local maximum to another one, which can happen when the optimization problem of agents is non-convex. In our example model, I did not encounter such a situation.

Each threshold point features three continuous variables: the location of the threshold $\hat{\xi}$, the left limit y^{left} of the continuous choice variable and the right limit y^{right} of the continuous choice variable at $\hat{\xi}$. To compute the first and second derivative of these quantities with respect to aggregate variables, one can apply the implicit function theorem on the system of three equations that determine the three variables. These equations are the following, where I use the notation $FOC(x, \hat{\xi}, y, d)$ for the first order condition that an interior solution y satisfies under discrete choice d, and $g(x, \hat{\xi}, y, d)$ denotes an occasionally binding constraint.

1. Hitting a boundary constraint, either of the endogenous state or of another endogenous variable (type 1. and type 2. above). At such a point, the policy function y is continuous, so that $y = y^{right} = y^{left}$, and we are left with two conditions:
   - the constraint is binding at the threshold:
     $g(x, \hat{\xi}, y(x, \hat{\xi}), d(x, \hat{\xi})) = 0$
   - the first order condition holds with equality at the threshold:
     $FOC(x, \hat{\xi}, y(x, \hat{\xi}), d(x, \hat{\xi})) = 0$
2. Change in a discrete choice variable d from $d = d_1$ to $d = d_2$ (type 3. above).
   - left limit of continuous choice is the optimum under d_1:
     $FOC(x, \hat{\xi}, y^{left}, d_1) = 0$
   - right limit of continuous choice is the optimum under d_2:
     $FOC(x, \hat{\xi}, y^{right}, d_2) = 0$
   - maximum achievable under d_1 equals maximum achievable under d_2:
     $v(x, \hat{\xi}, y^{left}, d_1) = v(x, \hat{\xi}, y^{right}, d_2)$
3. A discontinuity of the continuous choice variables appears at $\hat{\xi}$, although the discrete choice $d(x, \hat{\xi})$ does not change (type 3. above).

- left limit of continuous choice is a local optimum under d:
  $FOC(x, \hat{\xi}, y^{left}, d(x, \hat{\xi})) = 0$
- right limit of continuous choice is a local optimum under d:
  $FOC(x, \hat{\xi}, y^{right}, d(x, \hat{\xi})) = 0$
- y^{left} and y^{right} achieve the same value: $v(x, \hat{\xi}, y^{left}, d) = v(x, \hat{\xi}, y^{right}, d)$

As a practical matter, the identification of threshold points in the computation of the steady state is only done in the last steps of the iteration. In earlier stages, the integral of the value function over the distribution of the i.i.d. shock is computed by the standard quadrature rule, ignoring the kink at threshold points. This has only a very small impact on the overall value function, and it is sufficient to rectify this once the value function is close to convergence. In this way one also avoids anomalies, such as a household for which both the non-negativity of labor supply and the borrowing constraint are binding. In a normally calibrated model, this should not happen, because only rich people choose not to work, and only poor people are bound by the borrowing constraint. However, this can happen in the first period of the value function iteration, where every household saves the minimum amount.

C Finite Approximation of the Value and Distribution Functions

C.1 Value function approximation

The optimal policy depends through the Euler equation on the first derivative of the value function. The second derivative of the policy function then depends on the third derivative of the value function w.r.t. the individual continuous state, here household wealth. I approximate the value function by a cubic spline in household wealth. The third derivative of a cubic spline is discontinuous at the knot points, but is still right on average over any interval, if the second derivative is right at both ends of the interval. The discontinuity should therefore not have a significant effect on aggregates.

C.2 Distribution dynamics

The capital-dimension of the cross-sectional distribution is represented by the fraction of agents at the given set of grid points κ_i for $i = 1, \ldots, n_k$. Denote the end-of-period state by the "saving function" $K'(\kappa_i, \xi; \Omega)$ where Ω stands for all aggregate and individual states except k. Then we set the transition probabilities between grid points i and j as

$$\Pi_{i,j}(x) = prob[K'(\kappa_i, \xi; \Omega) \in (\bar{\kappa}_{j-1}, \bar{\kappa}_j)] \tag{55}$$

where the boundaries $\bar{\kappa}_j$ are chosen as $\bar{\kappa}_0 = \underline{k}$, $\bar{\kappa}_{n_k} = \infty$ and $\bar{\kappa}_j = (\kappa_j + \kappa_{j+1})/2$ for $j = 1, \ldots, n_k - 1$. Assuming that $K'(k, \xi; \Omega)$ increases monotonically in ξ, this can be written as

$$\Pi_{i,j}(x) = cdf(\Xi(\bar{\kappa}_j, \kappa_i; \Omega)) - cdf(\Xi(\underline{\kappa}_j, \kappa_i; \Omega)) \tag{56}$$

where the function Ξ is defined as

$$\Xi(k, \kappa_i; \Omega) \equiv \begin{cases} \underline{\xi} & \text{if } k \leq K'(\kappa_i, \underline{\xi}; \Omega) \\ \bar{\xi} & \text{if } k \geq K'(\kappa_i, \bar{\xi}; \Omega) \\ \xi \text{ s.t. } K'(\kappa_i, \xi; \Omega) = k & \text{else} \end{cases} \tag{57}$$

If $K'(k, \xi; \Omega)$ decreases monotonically in ξ, the formulas are reversed in the obvious way. Non-monotonic behavior would be more difficult to handle, first requiring to identify the maxima and minima of K'.

D Indivisible-Labor Model: Comparison to Existing Solutions

The model of indivisible labor in Chang and Kim (2007) is much harder to solve accurately than the model of divisible labor. Takahashi (2014) showed severe approximation errors in the original numerical results; improving the numerical approximation turns out to lead to substantially different results. The results presented below use the same calibration as Chang and Kim (2007), including the Markov process of productivity. The only substantial difference is the introduction of the smooth i.i.d. shock on labor productivity, ξ. Table 6 follows Tables 1 and 2 in Takahashi (2014), adding results of the linearized model solution with different levels of the standard deviation of ξ, $\sigma_\xi = 0.01, 0.03, 0.05$ as well as the quadratic model solution with $\sigma_\xi = 0.05$.

Table 6: Results Chang/Kim model

	Data	Chang/Kim	Takahashi	Lin,0.01	Lin,0.03	Lin,0.05	Quad,0.05
σ_H	0.82	0.76	0.57	0.54	0.53	0.52	0.52
σ_{wedge}	0.92	0.76	0.24	0.19	0.17	0.16	0.16
σ_Y	2.06	1.28	1.30	1.20	1.19	1.18	1.18
σ_C	0.45	0.39	0.33	0.33	0.33	0.33	0.33
σ_I	2.41	3.06	3.08	3.09	3.08	3.07	3.15
σ_L	-	0.50	0.41	0.39	0.37	0.37	0.37
$\sigma_{Y/H}$	0.50	0.50	0.49	0.50	0.51	0.51	0.51
$\rho(H, wedge)$	0.85	0.87	0.95	0.97	0.98	0.98	0.98
$\rho(H, Y/H)$	0.08	0.23	0.80	0.85	0.88	0.88	0.88
$\rho(H, wedge)$	0.85	0.87	0.95	0.97	0.98	0.98	0.98
$\rho(Y, C)$	0.69	0.84	0.89	0.90	0.90	0.90	0.90
$\rho(Y, I)$	0.90	0.98	0.99	0.99	0.99	0.99	0.97
$\rho(Y, H)$	0.86	0.87	0.96	0.97	0.97	0.97	0.97
$\rho(Y, L)$	—	0.92	0.95	0.97	0.97	0.98	0.98
$\rho(Y, Y/H)$	0.57	0.68	0.94	0.96	0.97	0.97	0.97

The table shows that the linearized and the quadratic solutions, in particular when σ_ξ is small, are very close to the Takahashi solution and clearly distinct from the Chang-Kim results. On the one hand, this is not surprising, because these solution methods do not suffer from the numerical problems pointed out by Takahashi. On the other hand, it is remarkable that perturbation approaches give results that are so close to Takahashi, who uses a version of the Krusell-Smith method.

E Details of Accuracy Checks

The following tables and graphs present the detailed results underlying the discussion in Section 4. For explanations, see Sections 4.1 – 4.3.

Table 7: Divisible-labor model, maximum error in impulse response to shock of 1 standard deviation

		Labor		Investment		Capital	
Reduc	#St.	Neg	NegPos	Neg	NegPos	Neg	NegPos
(ImpResp)		(3.7e-3)	(1.7e-5)	(5.8e-4)	(5.9e-7)	(4.9e-3)	(6.9e-6)
LIN	1400	1.69e-5	1.67e-5	5.85e-7	5.88e-7	6.88e-6	6.86e-6
-	0	1.28e-4	2.60e-6	4.93e-6	1.58e-7	2.66e-5	2.30e-6
CEA	1	9.20e-5	1.52e-6	2.75e-6	1.59e-7	1.80e-5	2.43e-6
CEA	2	2.34e-6	1.74e-6	2.11e-7	1.50e-7	2.85e-6	2.58e-6
CEA	3	1.57e-6	1.67e-6	1.78e-7	1.51e-7	2.74e-6	2.64e-6
CEA	4	1.75e-6	1.63e-6	1.56e-7	1.51e-7	2.64e-6	2.61e-6
CEA	6	1.75e-6	1.71e-6	1.56e-7	1.51e-7	2.74e-6	2.69e-6
CEA	8	1.70e-6	1.70e-6	1.56e-7	1.51e-7	2.75e-6	2.69e-6
CEA	10	1.70e-6	1.68e-6	1.56e-7	1.51e-7	2.78e-6	2.72e-6
MOM	1	1.12e-4	1.84e-6	3.45e-6	1.47e-7	2.26e-5	2.69e-6
MOM	2	7.88e-6	2.96e-6	2.19e-7	1.46e-7	3.73e-6	2.76e-6
MOM	3	8.83e-5	3.51e-5	2.56e-6	1.06e-6	2.79e-5	2.04e-5
MOM	4	1.15e-4	6.39e-5	4.41e-6	2.24e-6	2.52e-5	3.00e-5
PCA	1	4.61e-5	8.27e-7	1.27e-6	1.74e-7	1.19e-5	2.40e-6
PCA	2	6.47e-6	2.61e-6	4.66e-7	1.47e-7	6.13e-6	2.35e-6
PCA	3	6.15e-6	2.65e-6	3.93e-7	1.71e-7	3.84e-6	2.18e-6
PCA	4	6.68e-6	3.26e-6	3.65e-7	1.75e-7	3.71e-6	2.54e-6
PCA	6	4.57e-6	3.42e-6	2.48e-7	1.85e-7	3.20e-6	3.24e-6
PCA	8	3.82e-6	3.42e-6	2.22e-7	1.87e-7	3.50e-6	3.43e-6
PCA	10	3.57e-6	3.40e-6	2.44e-7	1.77e-7	3.07e-6	3.13e-6

Notes: Neg: response to negative shock; NegPos: sum of responses to negative and positive shock; #St: number of states added ot the minimal state vector; all solutions quadratic except "LIN"; row "(ImpResp)": maximum absolute impulse response;
cf. Section 4.2 for more explanations.

Table 8: Divisible-labor model, maximum error in impulse response to shock of 10 standard deviations

		Labor		Investment		Capital	
Reduc	#St.	Neg	NegPos	Neg	NegPos	Neg	NegPos
(ImpResp)		(3.9e-2)	(1.6e-3)	(5.8e-3)	(4.7e-5)	(4.8e-2)	(6.2e-4)
LIN	1400	1.68e-3	1.62e-3	4.71e-5	4.67e-5	6.19e-4	6.15e-4
-	0	1.46e-3	1.35e-4	5.04e-5	6.20e-6	2.81e-4	1.42e-4
CEA	1	1.07e-3	1.05e-4	2.80e-5	3.14e-6	1.47e-4	3.67e-5
CEA	2	8.52e-5	1.49e-5	1.75e-6	5.55e-7	9.76e-6	1.41e-5
CEA	3	6.99e-5	1.44e-5	1.67e-6	5.09e-7	6.88e-6	1.27e-5
CEA	4	7.01e-5	7.15e-6	1.65e-6	3.58e-7	5.65e-6	9.22e-6
CEA	6	6.98e-5	1.08e-5	1.67e-6	6.12e-7	1.01e-5	4.40e-6
CEA	8	6.97e-5	7.46e-6	1.69e-6	3.32e-7	1.56e-5	5.58e-6
CEA	10	6.98e-5	4.40e-6	1.67e-6	2.09e-7	1.10e-5	3.00e-6
MOM	1	1.28e-3	1.11e-4	3.44e-5	2.36e-6	2.20e-4	1.59e-5
MOM	2	1.44e-4	1.43e-4	4.42e-6	4.75e-6	7.49e-5	7.00e-5
MOM	3	2.77e-3	3.67e-3	8.91e-5	1.12e-4	1.42e-3	2.18e-3
MOM	4	1.03e-2	6.83e-3	3.73e-4	2.41e-4	4.98e-3	3.33e-3
PCA	1	4.92e-4	1.74e-4	1.37e-5	5.15e-6	6.57e-5	6.81e-5
PCA	2	9.73e-5	9.55e-5	7.58e-6	3.28e-6	9.72e-5	9.78e-5
PCA	3	1.12e-4	1.03e-4	8.01e-6	8.64e-6	2.22e-4	1.87e-4
PCA	4	1.82e-4	1.70e-4	1.27e-5	1.19e-5	2.60e-4	2.26e-4
PCA	6	2.35e-4	2.08e-4	1.44e-5	1.31e-5	3.19e-4	2.98e-4
PCA	8	2.36e-4	2.07e-4	1.43e-5	1.32e-5	3.57e-4	3.36e-4
PCA	10	2.24e-4	1.93e-4	1.20e-5	1.13e-5	3.30e-4	3.10e-4

Notes: Neg: response to negative shock; NegPos: sum of responses to negative and positive shock; #St: number of states added ot the minimal state vector; all solutions quadratic except "LIN"; row "(ImpResp)": maximum absolute impulse response;
cf. Section 4.2 for more explanations.

Table 9: Indivisible-labor model, maximum error in impulse response to shock of 1 standard deviation1

		Labor		Investment		Capital	
Reduc	#St.	Neg	NegPos	Neg	NegPos	Neg	NegPos
(ImpResp)		(3.4e-3)	(2.0e-5)	(7.0e-4)	(5.2e-7)	(5.5e-3)	(6.5e-6)
LIN	8500	2.01e-5	1.96e-5	4.08e-7	5.23e-7	4.57e-6	6.47e-6
-	0	6.72e-5	4.46e-6	3.43e-6	9.93e-8	1.34e-5	7.30e-7
CEA	1	6.44e-5	6.82e-6	3.33e-6	1.43e-7	1.08e-5	2.64e-6
CEA	2	7.99e-6	4.63e-6	8.83e-7	6.28e-8	2.19e-6	7.79e-7
CEA	3	2.85e-6	5.00e-6	4.92e-7	6.80e-8	1.92e-6	8.29e-7
CEA	4	2.39e-6	5.12e-6	2.33e-7	8.33e-8	1.37e-6	1.03e-6
CEA	6	2.45e-6	5.00e-6	2.47e-7	7.88e-8	1.57e-6	8.57e-7
CEA	8	1.21e-6	4.10e-6	2.24e-7	7.56e-8	1.64e-6	7.48e-7
CEA	10	1.03e-6	4.06e-6	2.35e-7	7.49e-8	1.72e-6	7.87e-7
MOM	1	1.17e-5	3.26e-6	1.04e-6	7.94e-8	3.57e-6	1.12e-6
MOM	2	1.05e-5	7.82e-6	7.56e-7	1.55e-7	3.86e-6	1.57e-6
MOM	3	1.45e-5	8.68e-6	7.45e-7	1.83e-7	2.76e-6	1.69e-6
MOM	4	9.40e-6	1.26e-5	7.82e-7	3.23e-7	5.56e-6	4.52e-6
PCA	1	1.46e-5	4.61e-6	5.41e-7	7.92e-8	6.97e-6	6.80e-7
PCA	2	9.29e-6	3.83e-6	4.96e-7	1.08e-7	4.22e-6	1.01e-6
PCA	3	5.88e-6	3.48e-6	4.06e-7	1.05e-7	3.59e-6	1.11e-6
PCA	4	3.84e-6	2.37e-6	3.57e-7	1.52e-7	4.00e-6	2.00e-6
PCA	6	3.47e-6	3.01e-6	3.62e-7	1.06e-7	3.43e-6	1.08e-6
PCA	8	1.02e-5	1.03e-5	3.78e-7	3.69e-7	9.95e-6	1.07e-5
PCA	10	1.27e-5	1.32e-5	4.16e-7	3.96e-7	1.13e-5	1.15e-5

Notes: Neg: response to negative shock; NegPos: sum of responses to negative and positive shock; #St: number of states added ot the minimal state vector; all solutions quadratic except "LIN"; row "(ImpResp)": maximum absolute impulse response;
cf. Section 4.2 for more explanations.

Table 10: Indivisible-labor model, maximum error in impulse response to shock of 10 standard deviations

		Labor		Investment		Capital	
Reduc	#St.	Neg	NegPos	Neg	NegPos	Neg	NegPos
(ImpResp)		(3.6e-2)	(2.1e-3)	(7.0e-3)	(5.1e-5)	(5.4e-2)	(6.4e-4)
LIN	8500	2.20e-3	2.13e-3	5.05e-5	5.15e-5	6.31e-4	6.39e-4
-	0	1.01e-3	2.66e-4	3.70e-5	7.06e-6	1.17e-4	6.21e-5
CEA	1	9.44e-4	2.70e-4	3.12e-5	9.86e-6	2.93e-4	2.84e-4
CEA	2	2.39e-4	6.39e-5	9.84e-6	7.14e-6	4.56e-5	4.57e-5
CEA	3	1.93e-4	8.42e-5	1.04e-5	7.66e-6	5.45e-5	5.34e-5
CEA	4	1.61e-4	9.20e-5	9.85e-6	7.12e-6	8.12e-5	9.94e-5
CEA	6	1.79e-4	9.54e-5	8.64e-6	5.91e-6	5.29e-5	8.25e-5
CEA	8	1.61e-4	5.19e-5	8.21e-6	5.48e-6	2.74e-5	3.55e-5
CEA	10	1.71e-4	5.75e-5	7.85e-6	5.12e-6	1.58e-5	3.21e-5
MOM	1	3.08e-4	2.36e-4	9.90e-6	7.89e-6	1.25e-4	9.88e-5
MOM	2	3.75e-4	3.60e-4	1.92e-5	1.64e-5	1.37e-4	1.50e-4
MOM	3	6.09e-4	7.44e-4	2.20e-5	1.92e-5	1.57e-4	2.19e-4
MOM	4	7.50e-4	8.14e-4	3.59e-5	3.32e-5	3.79e-4	3.94e-4
PCA	1	4.32e-4	2.04e-4	1.14e-5	6.16e-6	7.01e-5	7.55e-5
PCA	2	5.47e-4	3.70e-4	1.75e-5	1.16e-5	1.55e-4	1.12e-4
PCA	3	4.80e-4	3.57e-4	1.58e-5	1.15e-5	1.64e-4	1.22e-4
PCA	4	5.99e-4	4.70e-4	2.13e-5	1.61e-5	2.51e-4	1.98e-4
PCA	6	5.06e-4	3.76e-4	1.64e-5	1.05e-5	1.79e-4	1.20e-4
PCA	8	7.17e-4	6.97e-4	3.42e-5	3.48e-5	1.06e-3	1.10e-3
PCA	10	9.37e-4	9.45e-4	3.72e-5	3.92e-5	1.14e-3	1.17e-3

Notes: Neg: response to negative shock; NegPos: sum of responses to negative and positive shock; #St: number of states added ot the minimal state vector; all solutions quadratic except "LIN"; row "(ImpResp)": maximum absolute impulse response;
cf. Section 4.2 for more explanations.

Table 11: OLG model, maximum error in impulse response to simultaneous shocks of 3 standard deviations

		Labor		Investment		Capital	
Reduc	#St.	Neg	NegPos	Neg	NegPos	Neg	NegPos
(ImpResp)		(1.3e-2)	(5.6e-3)	(1.3e-2)	(2.9e-4)	(2.1e-2)	(1.5e-3)
LIN	16800	5.64e-3	5.57e-3	2.91e-4	2.89e-4	1.45e-3	1.45e-3
-	0	8.03e-4	3.26e-4	9.31e-5	1.44e-5	8.61e-4	1.01e-4
CEA	1	4.88e-4	1.36e-4	3.20e-5	1.09e-5	2.74e-4	6.25e-5
CEA	2	5.68e-4	1.30e-4	3.09e-5	1.03e-5	2.50e-4	6.45e-5
CEA	3	2.29e-4	1.41e-4	1.55e-5	1.14e-5	8.83e-5	5.79e-5
CEA	4	2.31e-4	1.40e-4	1.53e-5	1.13e-5	6.52e-5	3.48e-5
CEA	6	2.07e-4	1.41e-4	1.54e-5	1.13e-5	2.00e-5	1.46e-5
CEA	8	2.08e-4	1.41e-4	1.55e-5	1.14e-5	1.55e-5	1.14e-5
CEA	10	2.08e-4	1.41e-4	1.55e-5	1.14e-5	2.71e-5	1.14e-5
CEA	15	2.08e-4	1.41e-4	1.55e-5	1.14e-5	3.07e-5	1.15e-5
CEA	20	2.08e-4	1.42e-4	1.55e-5	1.14e-5	2.92e-5	1.14e-5
COH	3	5.26e-4	1.25e-4	2.56e-5	9.86e-6	3.20e-4	9.75e-5
COH	5	4.80e-4	3.60e-4	2.23e-5	1.09e-5	1.14e-4	1.49e-4
COH	11	3.50e-3	3.68e-3	1.82e-4	1.91e-4	1.42e-3	1.81e-3
PCA	1	5.41e-4	3.50e-4	1.02e-4	9.22e-6	6.42e-4	3.65e-5
PCA	2	7.23e-4	5.32e-4	5.77e-5	4.40e-5	1.25e-3	9.37e-4
PCA	3	7.41e-4	3.73e-4	7.03e-5	5.74e-5	1.44e-3	1.21e-3
PCA	4	4.21e-4	3.80e-4	6.37e-5	5.39e-5	1.41e-3	1.21e-3
PCA	6	1.03e-3	1.06e-3	6.64e-5	6.96e-5	1.22e-3	1.14e-3
PCA	8	1.23e-3	1.17e-3	6.22e-5	7.09e-5	1.20e-3	1.13e-3
PCA	10	1.03e-3	1.09e-3	5.42e-5	6.33e-5	1.17e-3	1.08e-3
PCA	15	1.11e-3	1.17e-3	4.76e-5	4.98e-5	6.63e-4	6.96e-4
PCA	20	1.67e-3	1.68e-3	7.04e-5	7.15e-5	7.18e-4	7.33e-4

Notes: Neg: response to negative shock; NegPos: sum of responses to negative and positive shock; #St: number of states added ot the minimal state vector; all solutions quadratic except "LIN"; row "(ImpResp)": maximum absolute impulse response;
cf. Section 4.2 for more explanations.

Table 12: OLG model, maximum error in impulse response to simultaneous shocks of 10 standard deviations

		Labor		Investment		Capital	
Reduc	#St.	Neg	NegPos	Neg	NegPos	Neg	NegPos
(ImpResp)		(1.4e-2)	(6.5e-2)	(4.6e-2)	(3.2e-3)	(7.7e-2)	(1.6e-2)
LIN	16800	6.55e-2	6.49e-2	3.25e-3	3.24e-3	1.61e-2	1.63e-2
-	0	5.72e-3	4.87e-3	2.84e-4	2.58e-4	3.02e-3	1.01e-3
CEA	1	5.13e-3	4.45e-3	1.79e-4	2.19e-4	7.66e-4	5.68e-4
CEA	2	5.04e-3	4.38e-3	1.72e-4	2.13e-4	7.31e-4	5.87e-4
CEA	3	5.16e-3	4.51e-3	1.44e-4	2.25e-4	9.53e-4	7.18e-4
CEA	4	5.14e-3	4.50e-3	1.43e-4	2.23e-4	2.90e-4	5.03e-4
CEA	6	5.15e-3	4.51e-3	1.43e-4	2.24e-4	4.08e-4	3.16e-4
CEA	8	5.16e-3	4.52e-3	1.44e-4	2.25e-4	2.06e-4	2.36e-4
CEA	10	5.16e-3	4.51e-3	1.45e-4	2.25e-4	1.45e-3	2.41e-4
CEA	15	5.16e-3	4.52e-3	1.47e-4	2.25e-4	1.54e-3	2.52e-4
CEA	20	5.16e-3	4.52e-3	1.44e-4	2.25e-4	1.34e-3	2.59e-4
COH	3	4.96e-3	4.33e-3	1.26e-4	2.08e-4	1.60e-3	1.15e-3
COH	5	4.60e-3	3.96e-3	1.39e-4	1.73e-4	1.39e-3	1.52e-3
COH	11	3.48e-2	4.21e-2	1.84e-3	2.19e-3	9.66e-3	2.32e-2
PCA	1	4.97e-3	4.25e-3	3.54e-4	2.01e-4	2.09e-3	5.22e-4
PCA	2	3.70e-3	6.38e-3	7.58e-4	5.40e-4	1.54e-2	1.22e-2
PCA	3	2.58e-3	3.33e-3	8.28e-4	7.24e-4	1.89e-2	1.59e-2
PCA	4	3.70e-3	2.42e-3	7.19e-4	6.81e-4	1.74e-2	1.54e-2
PCA	6	1.04e-2	1.19e-2	6.53e-4	7.84e-4	1.41e-2	1.50e-2
PCA	8	1.07e-2	1.27e-2	5.47e-4	7.61e-4	1.36e-2	1.48e-2
PCA	10	1.08e-2	1.31e-2	5.27e-4	7.42e-4	1.27e-2	1.37e-2
PCA	15	1.01e-2	1.21e-2	4.66e-4	5.93e-4	7.72e-3	8.89e-3
PCA	20	1.44e-2	1.41e-2	6.70e-4	6.87e-4	7.39e-3	9.06e-3

Notes: Neg: response to negative shock; NegPos: sum of responses to negative and positive shock; #St: number of states added ot the minimal state vector; all solutions quadratic except "LIN"; row "(ImpResp)": maximum absolute impulse response;
cf. Section 4.2 for more explanations.

Figure 2: Divisible-labor model, max. error to impulse response, 1 stdev

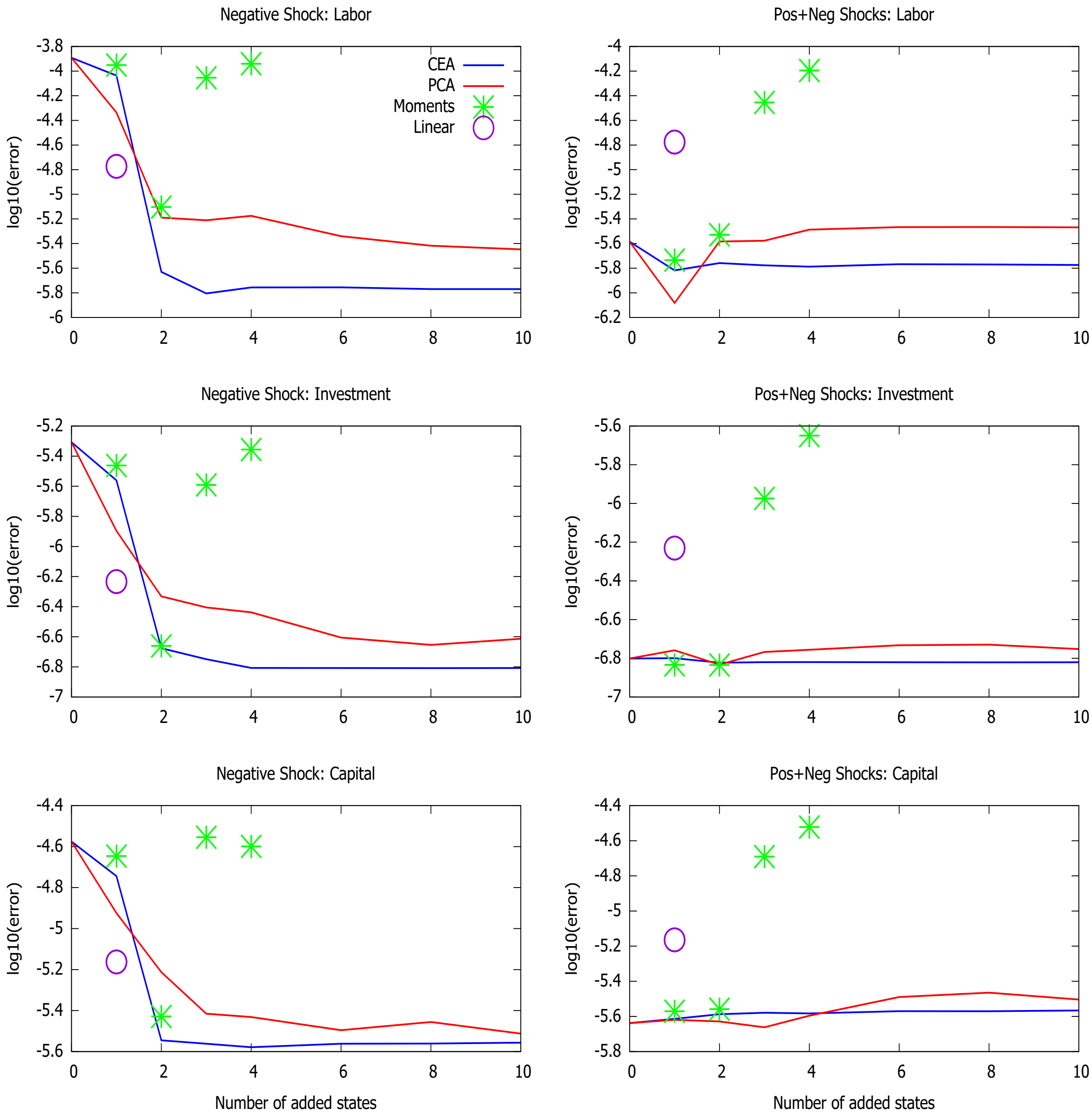

Figure 3: Divisible-labor model, max. error to impulse response, 10 stdev

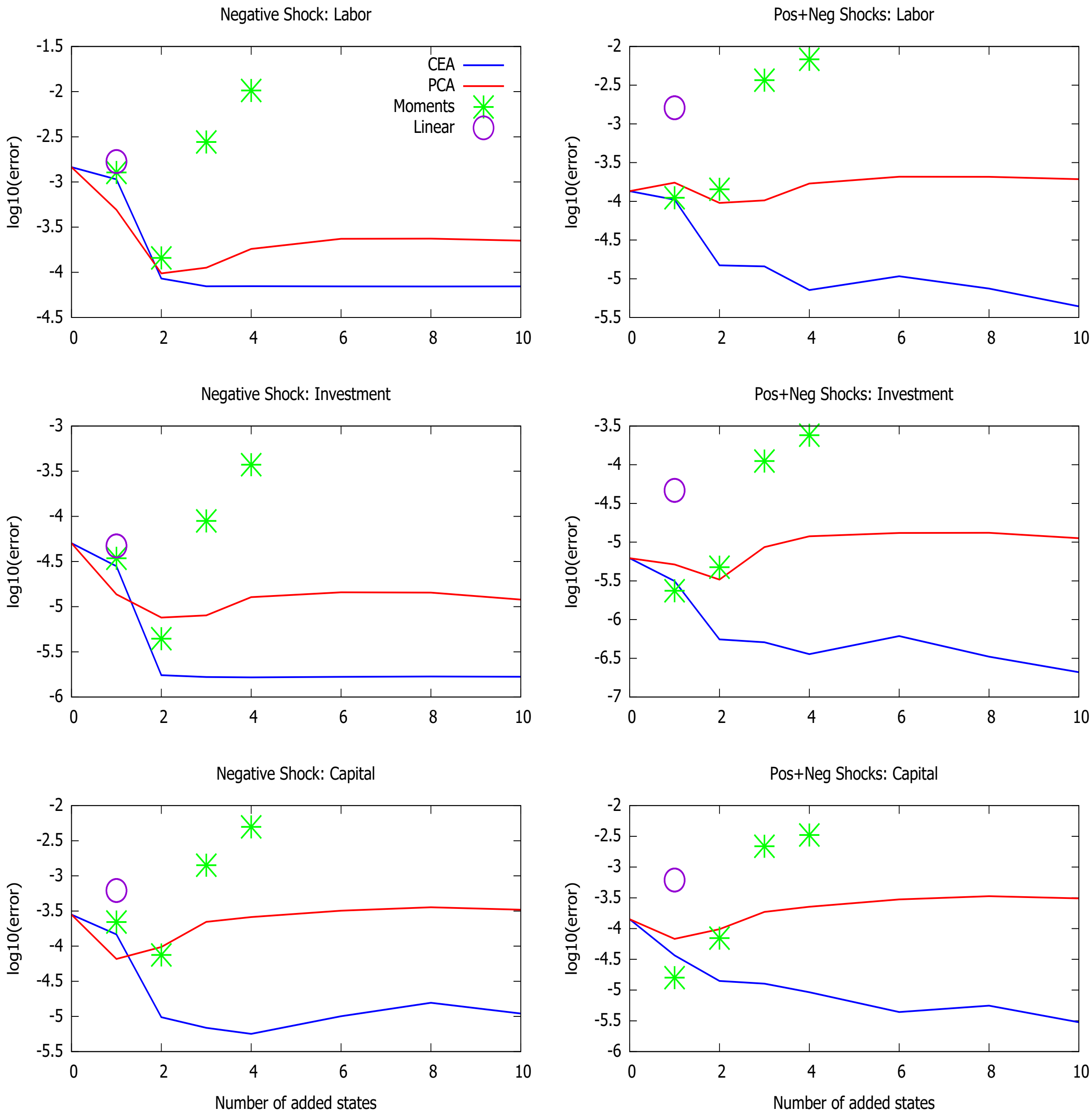

Figure 4: Indivisible-labor model, max. error to impulse response, 1 stdev

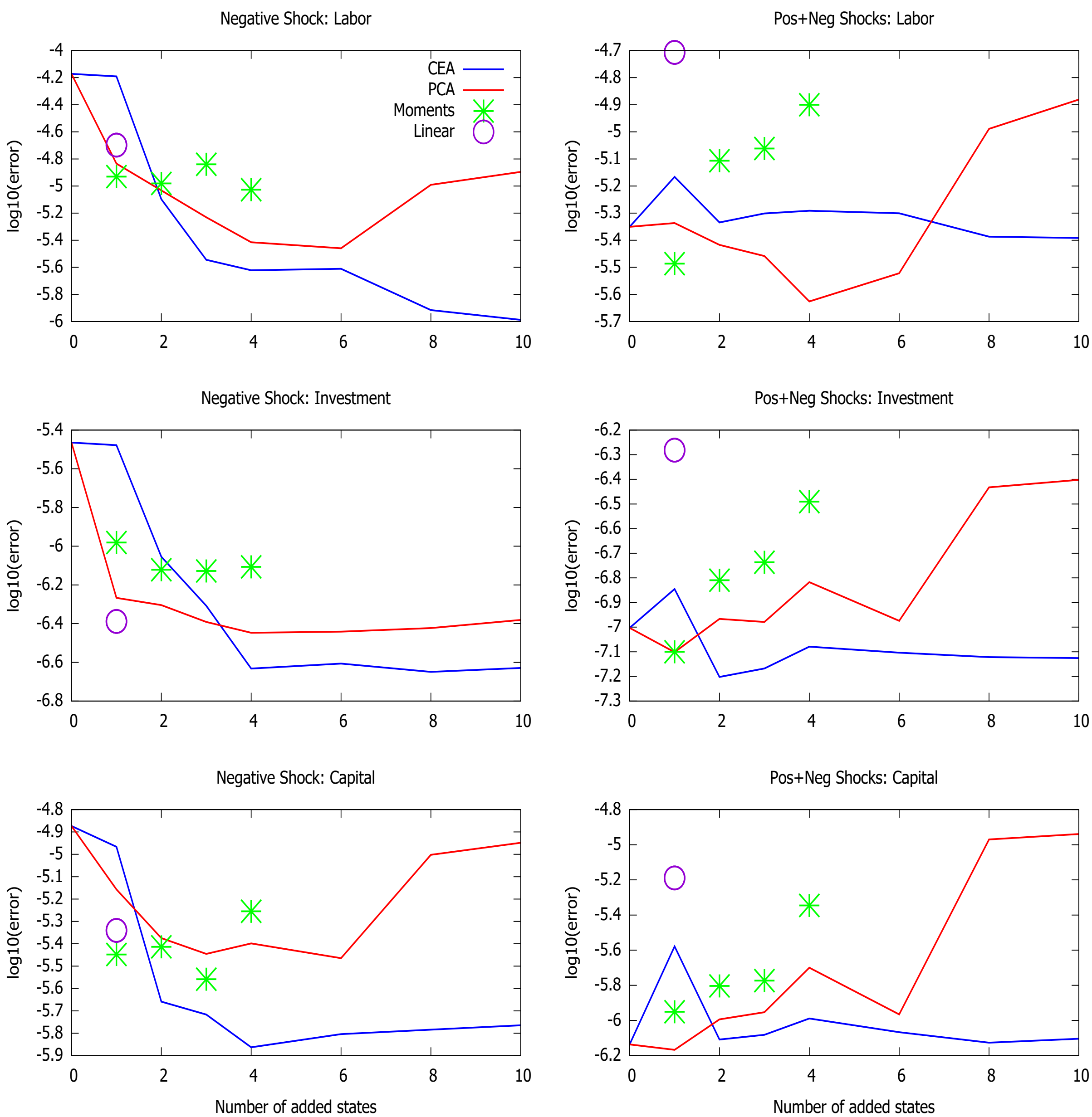

Figure 5: Indivisible-labor model, max. error to impulse response, 10 stdev

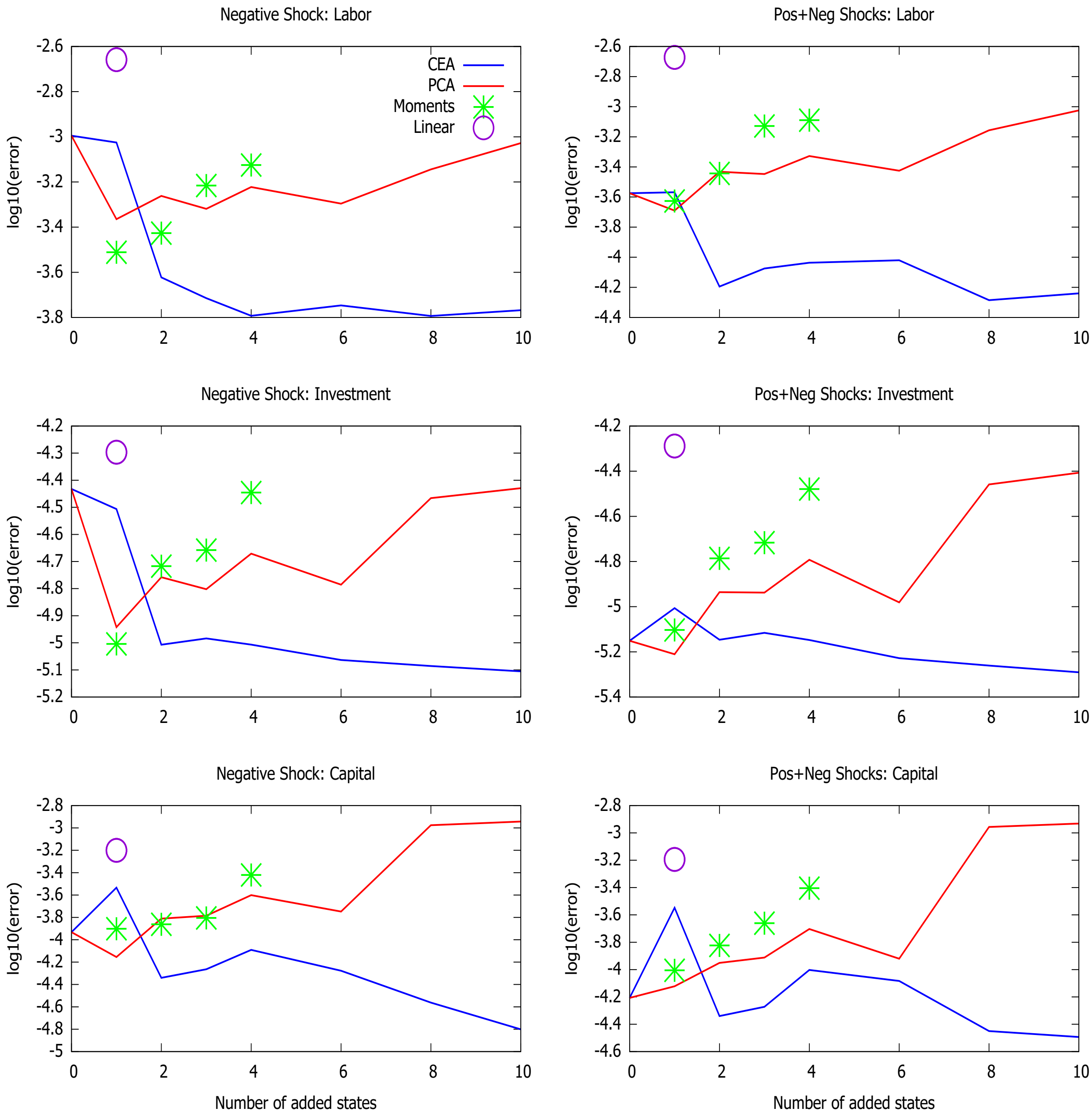

Figure 6: OLG model, max. error to impulse response all shocks, 3 stdev

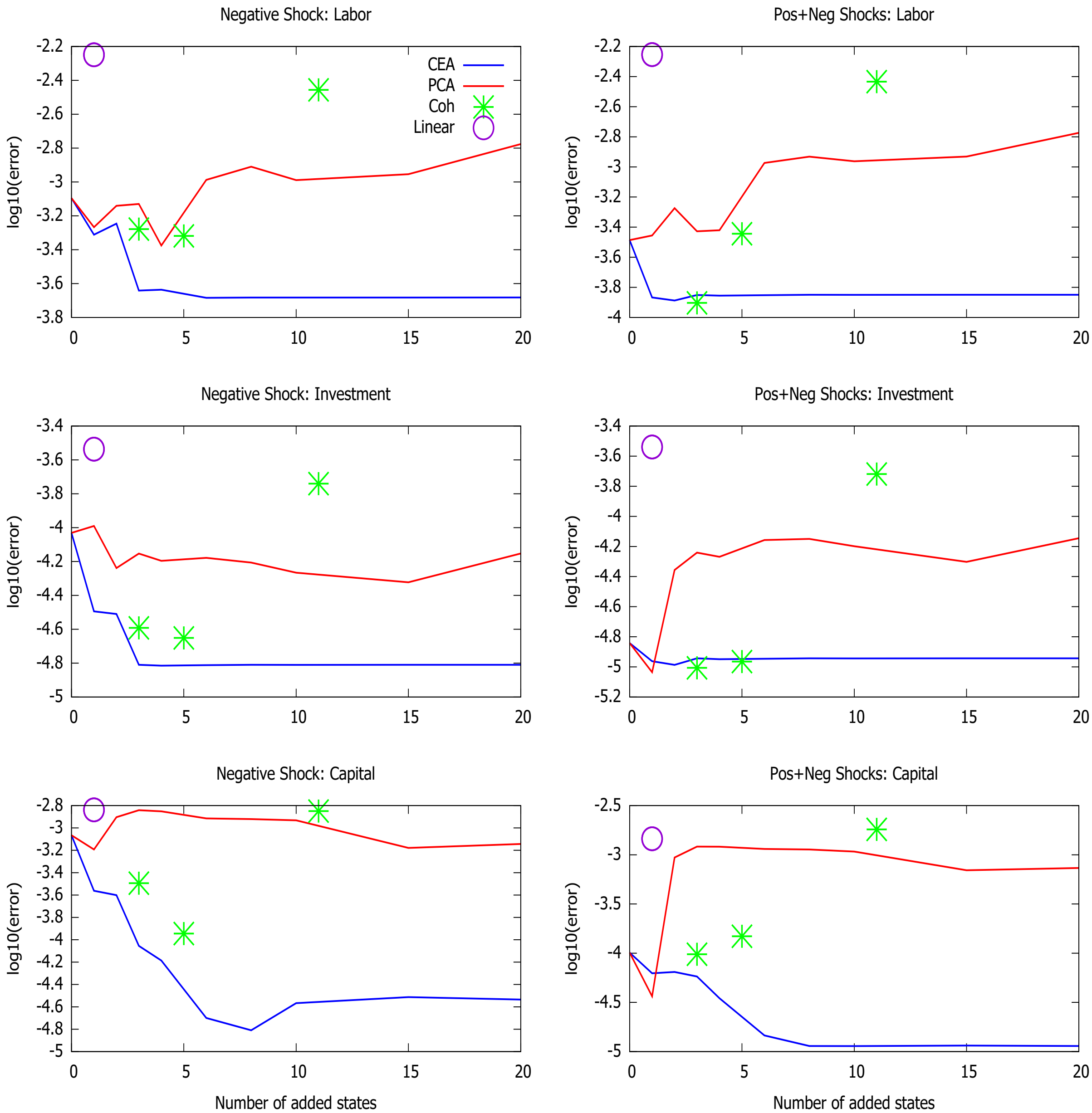

Figure 7: OLG model, max. error to impulse response all shocks, 10 stdev

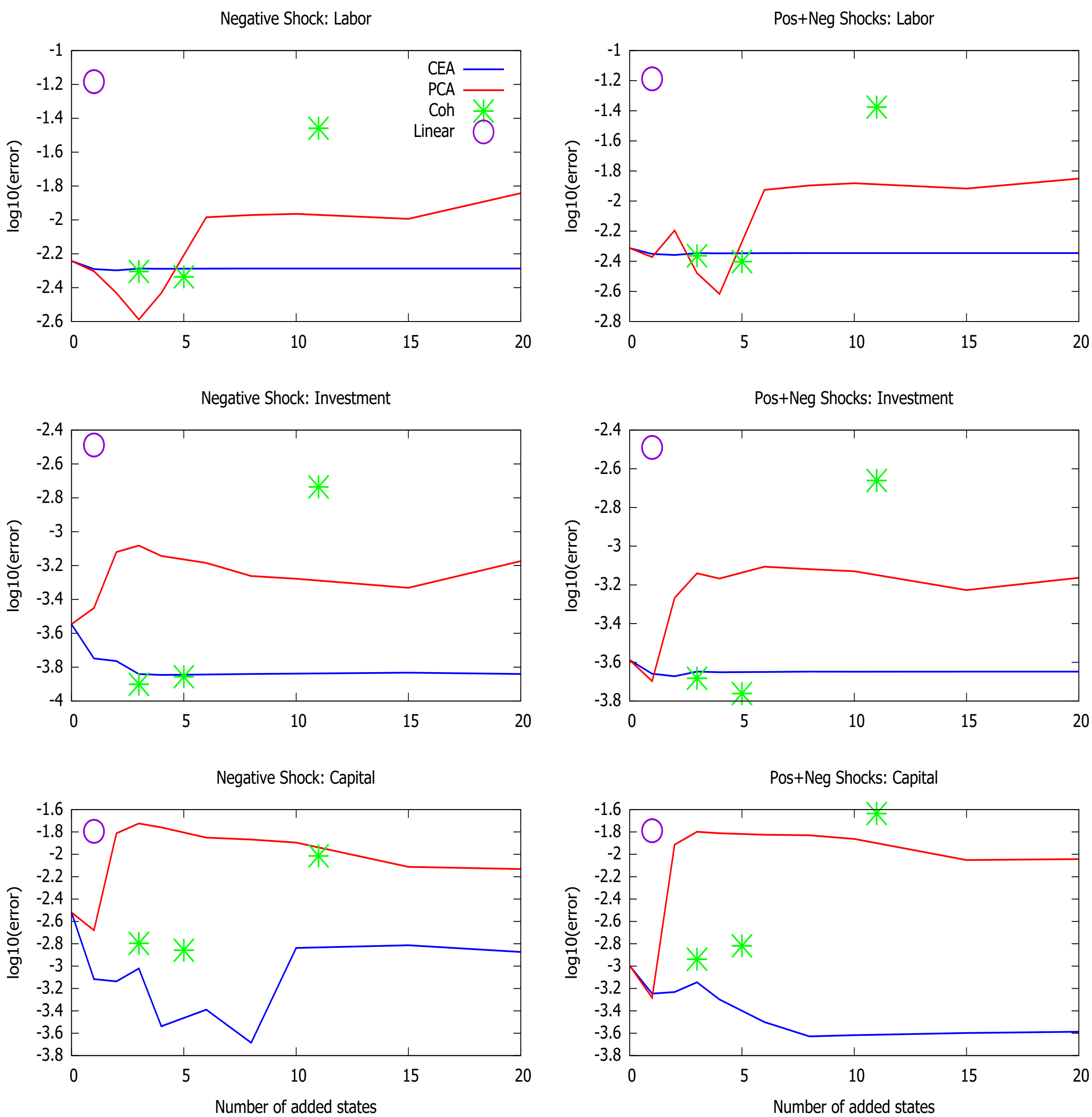

Table 13: OLG model, precautionary effects and Euler residual errors in simulation

		Precautionary effects			RMSE		
Reduc	#St	Labor	Cons.	Capital	Labor	Cons.	Capital
-	0	1.32e-4	-2.43e-4	6.27e-4	5.32e-4	4.69e-3	4.83e-3
CEA	1	1.93e-4	-4.53e-4	1.01e-3	4.92e-4	6.88e-4	1.78e-3
CEA	2	1.98e-4	-4.71e-4	1.04e-3	5.11e-4	3.16e-4	1.78e-3
CEA	3	1.66e-4	-3.62e-4	8.43e-4	2.25e-4	2.09e-4	8.85e-4
CEA	4	1.71e-4	-3.82e-4	8.78e-4	1.95e-4	1.82e-4	8.31e-4
CEA	6	1.71e-4	-3.82e-4	8.78e-4	1.43e-4	1.88e-4	7.65e-4
CEA	8	1.71e-4	-3.80e-4	8.75e-4	8.92e-5	1.52e-4	7.19e-4
CEA	10	1.71e-4	-3.81e-4	8.77e-4	2.15e-4	1.47e-4	9.96e-4
CEA	15	1.71e-4	-3.80e-4	8.75e-4	3.73e-4	1.11e-4	1.39e-3
CEA	20	1.71e-4	-3.80e-4	8.75e-4	4.51e-4	1.09e-4	1.59e-3
COH	3	1.83e-4	-4.21e-4	9.52e-4	4.29e-4	4.72e-4	1.68e-3
COH	5	2.39e-4	-6.15e-4	1.31e-3	7.98e-4	6.98e-4	2.13e-3
COH	11	4.37e-4	-1.27e-3	2.53e-3	3.02e-3	2.49e-3	8.08e-3
PCA	1	2.17e-4	-5.38e-4	1.17e-3	5.47e-4	2.61e-3	2.99e-3
PCA	2	8.09e-4	-2.63e-3	4.97e-3	5.37e-4	2.38e-3	2.47e-3
PCA	3	7.44e-4	-2.41e-3	4.56e-3	4.82e-4	2.29e-3	3.18e-3
PCA	4	6.83e-4	-2.20e-3	4.18e-3	5.00e-4	2.25e-3	3.24e-3
PCA	6	5.02e-4	-1.58e-3	3.03e-3	8.40e-4	2.03e-3	4.17e-3
PCA	8	4.76e-4	-1.49e-3	2.87e-3	8.56e-4	2.11e-3	4.23e-3
PCA	10	4.54e-4	-1.41e-3	2.73e-3	9.03e-4	2.10e-3	3.94e-3
PCA	15	2.53e-4	-6.93e-4	1.43e-3	7.66e-4	1.38e-3	3.44e-3
PCA	20	9.73e-5	-1.41e-4	4.23e-4	8.78e-4	8.74e-4	3.36e-3

Notes: #St: number of states added ot the minimal state vector; cf. Section 4.3 for detailed explanations.